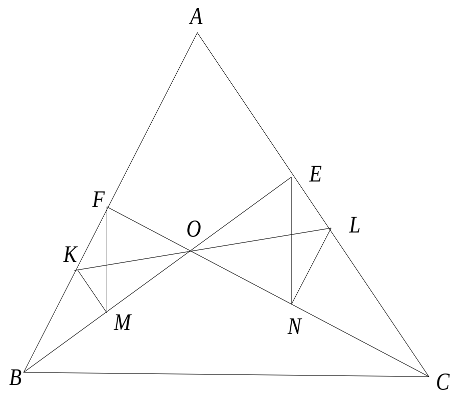
Solutions for problems of CRMO-2001

1. Let BE and CF be the altitudes of an acute triangle ABC, with E on AC and F on AB. Let O be the point of intersection of BE and CF. Take any line KL through O with K on AB and L on AC. Suppose M and N are located on BE and CF respectively, such that KM is perpendicular to BE and LN is perpendicular to CF. Prove that FM is parallel to EN.

   Solution: Observe that $KMOF$ and $ONLE$ are cyclic quadrilaterals. Hence

$$\angle FMO = \angle FKO, \quad \text{and} \quad \angle OEN = \angle OLN.$$



   However we see that

$$\angle OLN = \frac{\pi}{2} - \angle NOL = \frac{\pi}{2} - \angle KOF = \angle OKF.$$

   It follows that $\angle FMO = \angle OEN$. This forces that FM is parallel to EN.

2. Find all primes p and q such that $p^2 + 7pq + q^2$ is the square of an integer.

   Solution: Let p, q be primes such that $p^2 + 7pq + q^2 = m^2$ for some positive integer m. We write

$$5pq = m^2 - (p+q)^2 = (m+p+q)(m-p-q).$$

   We can immediately rule out the possibilities $m+p+q = p$, $m+p+q = q$ and $m+p+q = 5$ (In the last case $m > p, m > q$ and p, q are at least 2).

   Consider the case $m+p+q = 5p$ and $m-p-q = q$. Eliminating m, we obtain $2(p+q) = 5p-q$. It follows that $p = q$. Similarly, $m+p+q = 5q$ and $m-p-q = p$ leads to $p = q$. Finally taking $m+p+q = pq$, $m-p-q = 5$ and eliminating m, we obtain $2(p+q) = pq-5$. This can be reduced to $(p-2)(q-2) = 9$. Thus $p = q = 5$ or $(p, q) = (3, 11), (11, 3)$. Thus the set of solutions is

$$\{(p, p) \ : \ p \text{ is a prime }\} \cup \{(3, 11), (11, 3)\}.$$

3. Find the number of positive integers x which satisfy the condition

$$\left[\frac{x}{99}\right] = \left[\frac{x}{101}\right].$$

(Here $[z]$ denotes, for any real z, the largest integer not exceeding z; e.g. $[7/4] = 1$.)

Solution: We observe that $\left[\frac{x}{99}\right] = \left[\frac{x}{101}\right] = 0$ if and only if $x \in \{1, 2, 3, \ldots, 98\}$, and there are 98 such numbers. If we want $\left[\frac{x}{99}\right] = \left[\frac{x}{101}\right] = 1$, then x should lie in the set $\{101, 102, \ldots, 197\}$, which accounts for 97 numbers. In general, if we require $\left[\frac{x}{99}\right] = \left[\frac{x}{101}\right] = k$, where $k \geq 1$, then x must be in the set $\{101k, 101k+1, \ldots, 99(k+1)-1\}$, and there are $99 - 2k$ such numbers. Observe that this set is not empty only if $99(k+1) - 1 \geq 101k$ and this requirement is met only if $k \leq 49$. Thus the total number of positive integers x for which $\left[\frac{x}{99}\right] = \left[\frac{x}{101}\right]$ is given by

$$98 + \sum_{k=1}^{49}(99 - 2k) = 2499.$$

[**Remark:** For any $m \geq 2$ the number of positive integers x such that $\left[\frac{x}{m-1}\right] = \left[\frac{x}{m+1}\right]$ is $\frac{m^2-4}{4}$ if m is even and $\frac{m^2-5}{4}$ if m is odd.]

4. Consider an $n \times n$ array of numbers:

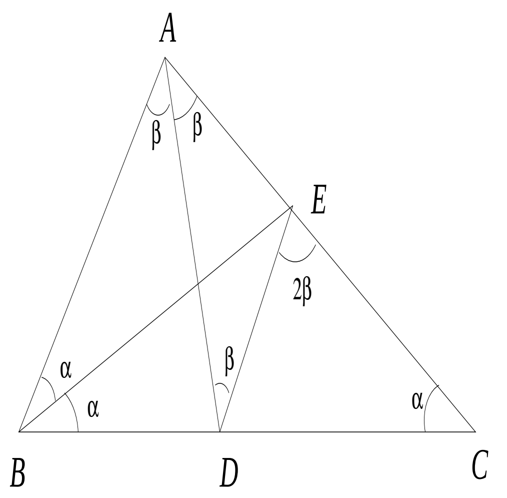
$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$$

Suppose each row consists of the n numbers $1, 2, 3, \ldots, n$ in some order and $a_{ij} = a_{ji}$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$. If n is odd, prove that the numbers $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$ are $1, 2, 3, \ldots, n$ in some order.

Solution: Let us see how many times a specific term, say 1, occurs in the matrix. Since 1 occurs once in each row, it occurs n times in the matrix. Now consider its occurrence off the main diagonal. For each occurrence of 1 below the diagonal, there is a corresponding occurrence above it, by the symmetry of the array. This accounts for an even number of occurrences of 1 off the diagonal. But 1 occurs exactly n times and n is odd. Thus 1 must occur at least once on the main diagonal. This is true of each of the numbers $1, 2, 3, \ldots, n$. But there are only n numbers on the diagonal. Thus each of $1, 2, 3, \ldots, n$ occurs exactly once on the main diagonal. This implies that $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$ is a permutation of $1, 2, 3, \ldots, n$.

5. In a triangle ABC, D is a point on BC such that AD is the internal bisector of $\angle A$. Suppose $\angle B = 2\angle C$ and $CD = AB$. Prove that $\angle A = 72°$.

Solution 1.: Draw the angle bisector BE of $\angle ABC$ to meet AC in E. Join ED. Since $\angle B = 2\angle C$, it follows that $\angle EBC = \angle ECB$. We obtain $EB = EC$.



Consider the triangles BEA and CED. We observe that $BA = CD$, $BE = CE$ and $\angle EBA = \angle ECD$. Hence $BEA \equiv CED$ giving $EA = ED$. If $\angle DAC = \beta$, then we obtain $\angle ADE = \beta$. Let I be the point of intersection of AD and BE. Now consider the triangles AIB and DIE. They are similar since $\angle BAI = \beta = \angle IDE$ and $\angle AIB = \angle DIE$. It follows that $\angle DEI = \angle ABI = \angle DBI$. Thus BDE is isoceles and $DB = DE = EA$. We also observe that $\angle CED = \angle EAD + \angle EDA = 2\beta = \angle A$. This implies that ED is parallel to AB. Since $BD = AE$, we conclude that $BC = AC$. In particular $\angle A = 2\angle C$. Thus the total angle of ABC is $5\angle C$ giving $\angle C = 36°$. We obtain $\angle A = 72°$.

Solution 2. We make use of the charectarisation: in a triangle ABC, $\angle B = 2\angle C$ if and only if $b^2 = c(c + a)$. Note that $CD = c$ and $BD = a - c$. Since AD is the angle bisector, we also have

$$\frac{a - c}{c} = \frac{c}{b}.$$

This gives $c^2 = ab - bc$ and hence $b^2 = ca + ab - bc$. It follows that $b(b + c) = a(b + c)$ so that $a = b$. Hence $\angle A = 2\angle C$ as well and we get $\angle C = 36°$. In turn $\angle A = 72°$.

6. If x, y, z are the sides of a triangle, then prove that

$$|x^2(y - z) + y^2(z - x) + z^2(x - y)| < xyz.$$

Solution: The given inequality may be written in the form

$$|(x - y)(y - z)(z - x)| < xyz.$$

Since x, y, z are the sides of a triangle, we know that $|x - y| < z$, $|y - z| < x$ and $|z - x| < y$. Multiplying these, we obtain the required inequality.

7. Prove that the product of the first 200 positive even integers differs from the product of the first 200 positive odd integers by a multiple of 401.

Solution: We have to prove that

$$401 \ \text{ divides } \ 2 \cdot 4 \cdot 6 \cdot \ \cdots \ \cdot 400 - 1 \cdot 3 \cdot 5 \cdot \ \cdots \ \cdot 399.$$

Write $x = 401$. Then this difference is equal to

$$(x-1)(x-3)\cdots(x-399) - 1 \cdot 3 \cdot 5 \cdot \ \cdots \ \cdot 399.$$

If we expand this as a polynomial in x, the constant terms get canceled as there are even number of odd factors ($(-1)^{200} = 1$). The remaining terms are integral multiples of x and hence the difference is a multiple of x. Thus 401 divides the above difference.