NATIONAL BOARD FOR HIGHER MATHEMATICS
AND
HOMI BHABHA CENTRE FOR SCIENCE EDUCATION
TATA INSTITUTE OF FUNDAMENTAL RESEARCH
Pre-REGIONAL MATHEMATICAL OLYMPIAD, 2012
Mumbai Region

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QUESTION PAPER SET: A

- There are 20 questions in this question paper. Each question carries 5 marks.
- Answer all questions.
- Time allotted: 2 hours.

QUESTIONS

1. Rama was asked by her teacher to subtract 3 from a certain number and then divide the result by 9. Instead, she subtracted 9 and then divided the result by 3. She got 43 as the answer. What would have been her answer if she had solved the problem correctly?

2. A triangle with perimeter 7 has integer side lengths. What is the maximum possible area of such a triangle?

3. For how many pairs of positive integers (x, y) is $x + 3y = 100$?

4. The letters R, M, and O represent whole numbers. If $R \times M \times O = 240$, $R \times O + M = 46$ and $R + M \times O = 64$, what is the value of $R + M + O$?

5. Let $S_n = n^2 + 20n + 12$, n a positive integer. What is the sum of all possible values of n for which S_n is a perfect square?

6. A postman has to deliver five letters to five different houses. Mischievously, he posts one letter through each door without looking to see if it is the correct address. In how many different ways could he do this so that exactly two of the five houses receive the correct letters?

7. In ΔABC, we have $AC = BC = 7$ and $AB = 2$. Suppose that D is a point on line AB such that B lies between A and D and $CD = 8$. What is the length of the segment BD?

8. In rectangle $ABCD$, $AB = 5$ and $BC = 3$. Points F and G are on line segment CD so that $DF = 1$ and $GC = 2$. Lines AF and BG intersect at E. What is the area of ΔAEB?

9. Suppose that $4^{X_1} = 5, 5^{X_2} = 6, 6^{X_3} = 7, \ldots, 126^{X_{123}} = 127, 127^{X_{124}} = 128$. What is the value of the product $X_1 X_2 \ldots X_{124}$?

10. $ABCD$ is a square and $AB = 1$. Equilateral triangles AYB and CXD are drawn such that X and Y are inside the square. What is the length of XY?

11. Let $P(n) = (n+1)(n+3)(n+5)(n+7)(n+9)$. What is the largest integer that is a divisor of $P(n)$ for all positive even integers n?

12. If $\dfrac{1}{\sqrt{2011 + \sqrt{2011^2 - 1}}} = \sqrt{m} - \sqrt{n}$, where m and n are positive integers, what is the value of $m + n$?

13. If $a = b - c$, $b = c - d$, $c = d - a$ and $abcd \neq 0$ then what is the value of $\dfrac{a}{b} + \dfrac{b}{c} + \dfrac{c}{d} + \dfrac{d}{a}$?

14. O and I are the circumcentre and incentre of ΔABC respectively. Suppose O lies in the interior of ΔABC and I lies on the circle passing through B, O, and C. What is the magnitude of $\angle BAC$ in degrees?

15. How many non-negative integral values of x satisfy the equation $\left[\dfrac{x}{5}\right] = \left[\dfrac{x}{7}\right]$?
(Here $[x]$ denotes the greatest integer less than or equal to x. For example $[3.4] = 3$ and $[-2.3] = -3$.)

16. Let N be the set of natural numbers. Suppose $f : N \to N$ is a function satisfying the following conditions.
    (a) $f(mn) = f(m)f(n)$;
    (b) $f(m) < f(n)$ if $m < n$;
    (c) $f(2) = 2$.

    What is the value of $\sum_{k=1}^{20} f(k)$?

17. Let x_1, x_2, x_3 be the roots of the equation $x^3 + 3x + 5 = 0$. What is the value of the expression
$$\left(x_1 + \frac{1}{x_1}\right)\left(x_2 + \frac{1}{x_2}\right)\left(x_3 + \frac{1}{x_3}\right)?$$

18. What is the sum of the squares of the roots of the equation $x^2 - 7[x] + 5 = 0$?
(Here $[x]$ denotes the greatest integer less than or equal to x. For example $[3.4] = 3$ and $[-2.3] = -3$.)

19. How many integer pairs (x, y) satisfy $x^2 + 4y^2 - 2xy - 2x - 4y - 8 = 0$?

20. PS is a line segment of length 4 and O is the midpoint of PS. A semicircular arc is drawn with PS as diameter. Let X be the midpoint of this arc. Q and R are points on the arc PXS such that QR is parallel to PS and the semicircular arc drawn with QR as diameter is tangent to PS. What is the area of the region $QXROQ$ bounded by the two semicircular arcs?

END OF QUESTION PAPER