These are some notes (written by Tejaswi Navilarekallu) used at the International Mathematical Olympiad Training Camp (IMOTC) 2013 held in Mumbai during April-May, 2013.

1 Introduction

It is quite often that we see problems in Olympaids which involve making *moves* to change a given configuration. A typical problem would have a given initial configuration, rules that every move should follow, and a final desired configuration. The moves can be completely deterministic, or can depend on certain choices (like in *games*). The task might be, for example, to show that a final configuration cannot be achieved, or to count the number of moves required to get to a certain configuration.

In this note, we shall explore a plethora of such problems and discuss a particular technique of associating (semi-)invariant quantities to the configurations.

2 Simple *semi-invariants*

As we shall see below, a key concept in problems with moves is to associate a quantity, an *invariant* or a *semi-invariant*, to each possible configuration such that it changes "uniformly" with every move (or remains unchanged for that matter). Typical uniformity we will see is the (strict) monotonicity. If, for example, the semi-invariant is a positive integer that strictly decreases with every move, then by induction on this semi-invariant it follows that we can make only finitely many moves. This is a standard way to attack many problems and we shall see this being used repeatedly below. We shall sometime hide the semi-invariant and only consider the induction, but the ideas behind each of them will be similar.

Let us start with the following (very) simple example.

Example 2.1. Let n be a positive integer. Written on a board are n natural numbers $x_1, x_2, \ldots, x_n$. A *move* consists of replacing two numbers on the board by their sum (until there is only one number on the board). Then the number on the board at the end is independent of the choices of moves.

It is clear from the description of a move that the number of integers on the board reduces with every move. This is our first example of a semi-invariant! For a configuration C of a bunch of numbers on the board, we define $I(C)$ to be the number of integers in this configuration. If a move takes C to C' then $I(C') = I(C) - 1 < I(C)$. Therefore we will end up with one number after exactly $n-1$ moves. This sounds like a complicated way of making a simple observation, but we shall consider less trivial semi-invariants soon.

Consider a move that replaces two numbers x and y on the board by $x+y$. As mentioned above, we want to associate each configuration with a quantity that changes uniformly. Given that we are replacing the numbers by their sums, it is "natural" that the sum of the numbers on the board changes uniformly. In fact, the sum does not change at all. More precisely, for any configuration C, let $J(C)$ denote the sum of all the numbers on the board in that configuration. If a move takes C to C' then we have $J(C) = J(C')$. Therefore J is an invariant. Hence it follows that the number on the board at the end is $J(C_0) = x_1 + x_2 + \cdots + x_n$ (with C_0 the initial configuration), which is independent of the choices of moves.

That was a lengthy explanation for a very simple example. Let us make it a bit more interesting by changing the example slightly.

Example 2.2. Written on a board are the numbers $1, 2, 3, \ldots, 2013$. In a *move*, Alice can choose two numbers x and y on the board and replace them with $x - y$. Can Alice make 2012 moves in such a way that the final number on the board is 1000?

This example differs from the previous one only by a sign, i.e., we have $x - y$ instead of $x + y$ as part of the rule. Since we already know some nice invariant for the previous example, let us instead look for the similarity between the two. As a more general observation, the (only) similarity between $x-y$ and $x+y$ is their *parity*. That is, either both of them are odd, or both are even. This hints that if we consider the same quantity $J(C)$ as in the previous example, then even though it may not remain invariant under a move, its parity will. To be precise, for a configuration C of numbers on the board, let $J(C)$ denote the sum of all those numbers. Then $J(C) \pmod 2$ is invariant under a move. Since the sum of all the numbers on the board initially is odd and 1000 is even, it follows that the desired final configuration cannot be achieved.

Problem 2.3. Written on a blackboard are n nonnegative integers whose greatest common divisor is 1. A move consists of erasing two numbers x and y, where $x \geq y$, on the blackboard and replacing them with the numbers $x - y$ and $2y$. Determine for which original n-tuples of numbers on the blackboard is it possible to reach a point, after some number of moves, where $n - 1$ of the numbers on the blackboard are zeros.

Discussion. Let us start examining the semi-invariants. A move replaces two numbers x and y with $x - y$ and $2y$. What is evident at first sight is that the sum of the numbers remains the same after any move. There is another simple observation we can make. Note that we eventually want many zeros on the board. A move that erases two numbers x and y, with $y = 0$, will replace those numbers back, and hence it has no effect. Therefore any zero that appears on the board will remain forever. Thus the number of zeros on the board can never decrease, so the number of zeros on the board is a semi-invariant. This semi-invariant is a non-negative integer, increases monotonically with every move and is bounded above by n. Unfortunately, it is not a strictly increasing function, so we need something else.

Working out examples is in our opinion one of the best ways to get an insight. This is one of the few aspects that we shall repeat every now and then! When working out examples, it is most often the best to start with the simplest possible ones. In this problem, we can start with $n = 2$ and with numbers $1, a$ on the board. As we proceed with the first few values of a we can guess that the desired end point is reached if and only if $a = 2^k - 1$ for some $k \geq 0$. We can try to work out some more examples with $a, b > 1$ on the board to start with and we see in all the cases that $a + b$ is a power of two. If we now have more than two numbers on the board in the beginning, then to get to the desired end configuration we have to pass a stage in which there are exactly two non-zero numbers on the board. From what we have guessed so far, the sum of these two numbers should be a power of two. Since the sum of all the numbers on the board is an invariant our guess should be that $a_1 + a_2 + \cdots + a_n = 2^k$ for some $k \geq 0$, where $a_1, a_2, \ldots, a_n$ are the numbers on the board.

Let us go back to the case $n = 2$. If a and b are the numbers on the board in the beginning, then for their sum to be a positive power of 2 we need both of them to be of the same parity. They cannot both be even since they should be coprime to each other (by the given condition of the problem), and hence they are both odd. After the first move, we are left with two even numbers. It is evident that henceforth all the numbers that appear on the board will be even. We can therefore remove a factor of 2, or rather a factor of a power of 2, and reduce the case back to odd numbers.

Thus the power of two dividing the *gcd* of the two numbers increases (strictly) at every stage. This is an excellent semi-invariant! In fact, the same semi-invariant works in the general case. As we have observed that the *gcd* is what we need to consider, we note that either $(x-y, 2y) = (x, y)$ or $(x-y, 2y) = 2(x, y)$. This will be the key in the solution below. In the solution below, we shall not explicitly consider the semi-invariant, but it is there in a hidden form.

Solution. We shall show that we can reach the desired configuration if and only if the sum of all the numbers on the board is a power of two.

We make two observations. Firstly, the sum of all the numbers on the board is invariant under a move and secondly, $(x-y, 2y) = (x, y)$ or $(x-y, 2y) = 2(x, y)$. Since the *gcd* of the numbers to start with is 1 it follows that the *gcd* of the numbers on the board is a power of 2 after any number of moves. If we can reach a point where $n-1$ of the numbers are zero, then the non-zero number at this stage is a power of two. But, by the first observation, this is the sum of all the numbers at the begining. This proves the only if implication.

Our third observation is that if we have the numbers $a_1, a_2, \ldots, a_n$ on the board, then we can reach the desired configuration by making moves if and only if we can do the same with the numbers $a_1/d, a_2/d, \ldots, a_n/d$ on the board, where $d = (a_1, a_2, \ldots, a_n)$.

Now suppose that $a_1 \leq a_2 \leq \ldots \leq a_n$ are n non-negative integers on the board such that $a_1 + a_2 + \cdots + a_n = 2^k$, with $k \geq 0$. We shall show by induction on k that we can reach the desired configuration. The statement is obvious if $k = 0$. Suppose that $k > 0$. Then the number of odd numbers among $a_1, a_2, \ldots, a_n$ is even. Pair them up in any order and for each pair (x, y) with $x \geq y$, replace them with $x - y$ and $2y$ (i.e., make a move with these numbers). Note that both $x - y$ and $2y$ are even, and therefore we will end up with all even numbers, say $b_1, b_2, \ldots, b_n$. Since $b_1/2 + b_2/2 + \cdots + b_n/2 = 2^{k-1}$ it follows by the induction hypothesis that we can reach the desired configuration starting with the numbers $b_1/2, b_2/2, \ldots, b_n/2$. By the third observation, we can therefore reach the desired configuration starting with the numbers $b_1, b_2, \ldots, b_n$. This completes the induction step and concludes the solution. □

Problem 2.4. Let M be a set of six distinct positive integers whose sum is 60. These numbers are written on the faces of a cube, one number to each face. A move consists of choosing three faces of the cube that share a common vertex and adding 1 to the numbers on those faces. Determine the number of sets for which it is possible, after a finite number of moves, to produce a cube all of whose sides have the same number.

Discussion. Note that the desired configuration in this problem is not an *end* configuration since we can get different configurations if we make more moves thereafter. So we are not really interested in a semi-invariant, but rather we are looking for some invariant. Let us make the notation a bit more precise, and we shall see immediately what we need to look for.

We let $i, j = 1, 2, 3$ in this entire paragraph. Let A_i and B_i denote the opposite faces of the cube and let a_i and b_i denote the numbers on them respectively. Then a move consists of choosing one face from each of $\{A_i, B_i\}$ and increasing the number on it by 1. So for any i, the sum $a_i + b_i$ will increase by 1. Therefore the difference of two such sums is an invariant. Thus to get the desired configuration it is necessary to have these sums equal.

For the converse, we have to make the right choices of the moves to make all the numbers equal. This is not very difficult – we can make the numbers on each of the opposite sides equal.

Solution. For $i = 1, 2, 3$, let A_i and B_i denote the opposite faces of the cube and let a_i and b_i denote the numbers on them respectively. Then for any $1 \le i, j \le 3$, the quantity $a_i + b_i - a_j - b_j$ is invariant under a move. Therefore, if after finite number of moves, all the sides have the same number then $a_i + b_i = 20$ for all $i = 1, 2, 3$. Thus $M = \{x, y, z, 20 - x, 20 - y, 20 - z\}$ where x, y, z are integers with $1 \le x < y < z \le 9$, and there are $\binom{9}{3} = 84$ such possible sets.

Conversely, suppose that M is one of these 84 sets. We may assume that $a_i \le b_i$ for $i = 1, 2, 3$ and that $b_1 - a_1 \le b_2 - a_2 \le b_3 - a_3$. We add $(b_1 - a_1)/2 + (b_3 - a_3)/2$ to the faces containing a_1, a_2 and a_3, $(b_2 - a_2)/2 - (b_1 - a_1)/2$ to the faces containing b_1, a_2 and a_3, and $(b_3 - a_3)/2 - (b_2 - a_2)/2$ to the faces containing b_1, b_2 and a_1. It is easy to see that all the numbers added are integers. With these moves we will have all the numbers equal to b_3. □

Problem 2.5 (USAMO 2003, Problem 6)**.** At the vertices of a regular hexagon are written six nonnegative integers whose sum is n. One is allowed to make moves of the following form: (s)he may pick a vertex and replace the number written there by the absolute value of the difference between the numbers written at the two neighbouring vertices. Prove that if n is odd then one can make a sequence of moves, after which the number 0 appears at all six vertices.

Discussion. There are two semi-invariants that we can think of at first sight: the number of zeros and the maximum of the six numbers. The former is monotonic only if we choose our moves carefully (otherwise this number could both increase and decrease). The latter however is a monotonically decreasing quantity. The only instance in which this latter quantity cannot be reduced further is when we have the numbers $a, a, 0, a, a, 0$ at the vertices (in that order), a deadlock situation. So if we somehow make sure that this configuration is never reached then we can get to all zeros. We cannot blindly keep reducing one of the largest numbers, because the configuration $a+1, a, 0, a, a, 0$ will then result in a deadlock. So we have to be bit more careful.

The given condition that n is odd is quite crucial (note that the deadlock situation has the sum even). One can therefore try to keep the sum of all numbers odd (unless all the numbers become zero). This may not be possible after every move, but we can group the moves together such that this is the case. Such grouping was also part of the solution to Problem 2.3, albeit in a hidden way.

To say a little bit more about such groupings, note that we have either one, three or five odd numbers to start with. It is a common idea to reduce many such possibilities to one which is *most suited* to the situation. In this case, there is no such configuration that is most suited. Some configurations might be more intuitive to one and some to the others. Here we give one reason why the solution below is chosen among many possible ones. Remember that we want to reduce the maximum value. If this maximum value is even and is adjacent to vertices with odd numbers, then we can reduce the value without changing the configuration type. For this purpose, it is good to have even and odd numbers at every alternate vertices. We therefore consider this as a special configuration and try to reduce every other configuration to this.

Solution. By an *odd configuration* C we mean six non-negative numbers $C = (a_1, a_2, \dots, a_6)$ such that $\sum_{i=1}^{6} a_i$ is odd. Let t_k denote the operation at vertex k, i.e., replacing a_k with $|a_{k-1} - a_{k+1}|$, with k written modulo 6. If $M(C)$ denotes the maximum of the six numbers, then note that application of t_k will not increase the value of M. We call an odd configuration C *very odd* if the six numbers are alternatively odd and even. The following two-step algorithm terminates since after every application of Step 2 to a configuration C we get a configuration C' with $M(C') < M(C)$ or $C' = (0, 0, 0, 0, 0, 0)$.

- **Step A.** Suppose that C is odd but not very odd. Suppose first that we have five odd numbers. Then we may assume that $C \equiv (1,1,1,1,1,0) \pmod 2$. Then apply t_2 and t_4 to get a very odd configuration.

  Suppose that we have three odd numbers, but they are not on the alternative vertices. Then there are two possibilities (after symmetry): $C \equiv (1,1,1,0,0,0)$ or $C \equiv (1,1,0,1,0,0)$ modulo 2. In the former case apply t_4 and t_6 to reduce to the case of five odd numbers, and in the latter case apply t_6 followed by t_1 to get a very odd configuration.

  Finally suppose that we have one odd number. Then we may assume that $C \equiv (1,0,0,0,0,0)$ $\pmod 2$. Then apply t_2 and t_6 to reduce to the previous case.

- **Step B.** Suppose that C is very odd, so $C \equiv (1,0,1,0,1,0) \pmod 2$. Applying t_2, t_4 and t_6 if necessary, we may assume that $M(C)$ is odd. If $M(C) \neq a_1$ then apply t_3 and t_5 to get an odd configuration C' with $M(C') < M(C)$, and then go to Step A. We take similar actions if $M(C) \neq a_3$ or $M(C) \neq a_5$. If $M(C) = a_1 = a_3 = a_5$, the apply t_2, t_4, t_6 followed by t_1, t_3, t_5 to get $(0,0,0,0,0,0)$ and terminate the algorithm.

□

3 Moving on with moves

We have seen a few problems with simple semi-invariants. We now move on to slightly more involved quantities. Some of these are not very easy to find, but we shall give some intuitive arguments towards obtaining them.

Problem 3.1 (IMO 1986, Problem 3)**.** To each vertex of a regular pentagon an integer is assigned so that the sum of all five numbers is positive. If three consecutive vertices are assigned the numbers x, y, z respectively, and $y < 0$, then the following operation is allowed: x, y, z are replaced by $x+y, -y, z+y$ respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure necessarily comes to an end after a finite number of steps.

Discussion. Considering a few examples with small numbers, one should quickly guess that the procedure necessarily comes to an end after a finite number of steps. It is then a question of assigning a semi-invariant that changes monotonically with every operation. There are many possible choices for this. First let us look at some quantities that fail to serve the purpose. Before anything else, note that the sum of all the numbers is an invariant. A natural quantity to look at in such cases is the sum of all the squares. A simple calculation then tells us that this is not monotonic with respect to the operations.

The next (wrong) guess could be the minimum of the five numbers. If x, y, z are all negative, then we have bigger negative numbers than before. This leads to the intuition that if a negative number is next to a positive number then the value of the semi-invariant should be "better". In other words, the quantity should take into account the neighbours of every vertex. We can therefore consider $\sum |a_i + a_{i+1}|$ or $\sum (a_i + a_{i+1})^2$ where a_i's are the numbers at the vertices and the index i is taken modulo 5. After an operation, these quantities will have terms of the form $|a_i|$ (or a_i^2), and of the form $|a_i + a_{i+1} + a_{i+2}|$ or $(a_i + a_{i+1} + a_{i+2})^2$. For the first quantity, to compensate for these extra terms we can consider $\sum(|a_i|+|a_i+a_{i+1}|+|a_i+a_{i+1}+a_{i+2}|+|a_i+a_{i+1}+a_{i+2}+a_{i+3}|)$. This is then a semi-invariant. For the second quantity, it is even simpler: we just consider $\sum a_i^2 + (a_i + a_{i+1})^2$. We give both these solutions below. [1]

Solution. Let $a_1, a_2, \ldots, a_5$ be the numbers at the vertices, in that order, and let $s > 0$ denote the sum of these five numbers. Let

$$I = I(a_1, a_2, \ldots, a_5) = \sum \left(|a_i| + |a_i + a_{i+1}| + |a_i + a_{i+1} + a_{i+2}| + |a_i + a_{i+1} + a_{i+2} + a_{i+3}| \right),$$

where all the sums are taken over $1 \leq i \leq 5$ and the indices are written modulo 5. Suppose that we perform an operation with $x = a_1, y = a_2$ and $z = a_3$. Then we get $I' = I(a_1 + a_2, -a_2, a_2 + a_3, a_4, a_5) = I - |s - a_2| + |s + a_2|$. Since $s > 0$ and $a_2 < 0$ it follows that $I' < I$. Thus I is a positive quantity that reduces with every operation. Therefore the procedure necessarily comes to an end. □

Alternate solution. With the notation as in the above solution, let

$$I = I(a_1, a_2, \ldots, a_5) = \sum a_i^2 + \sum (a_i + a_{i+1})^2 .$$

Then $I' = I(a_1 + a_2, -a_2, a_2 + a_3, a_4, a_5) = I + 2a_2(s - a_2) < I$. Thus the quantity I is a positive integer that reduces with every operation and hence it follows that the procedure necessarily comes to an end. □

Problem 3.2 (USAMO 2011, Problem 2)**.** An integer is assigned to each vertex of a regular pentagon so that the sum of the five integers is 2011. A turn of a solitaire game consists of subtracting an integer m from each of the integers at two neighboring vertices and adding $2m$ to the opposite vertex, which is not adjacent to either of the first two vertices. (The amount m and the vertices chosen can vary from turn to turn.) The game is won at a certain vertex if, after some number of turns, that vertex has the number 2011 and the other four vertices have the number 0. Prove that for any choice of the initial integers, there is exactly one vertex at which the game can be won.

Discussion. This is another problem where the sum of all the numbers is an invariant. The aim of this problem is a bit different in comparison with the ones we have discussed so far. In this case, we are looking for an invariant that associates a vertex to every configuration. Since there are five vertices, this invariant can be thought of as an integer modulo 5, each corresponding to a vertex. In particular, the vertices are to be numbered $0, 1, 2, 3, 4$. Note that any change in the numbering of these vertices should also change the invariant in a similar fashion. So the invariant should be a *weighted* sum of the numbers.

Let us look at the trivial example where the numbers are $1, 0, 0, 0, 0$. Then if the vertex to which 1 is assigned is numbered i, then the invariant should be i. Does that give any ideas?

On the other hand, consider a turn in which we add $2m$ to the integer at vertex v. The operation is *symmetric* with respect to this vertex. If we are looking for a weighted sum that is invariant under every turn, then the weight at v should equal to the sum of the weights at the other two vertices (at which subtraction takes place). Suppose the weight at v is a then the sum of the weights

[1] A third solution can be given by consider the semi-invariant $\sum(a_i - a_{i+2})^2$.

at the other two vetices should be $2a$. Intuitively, this leads to assigning weights $a-2$ and $a+2$ at these vertices. These ideas lead to the following solution.

One more point to note is that there are two parts to the given problem: to show that there is at most one vertex at which the game can be won; and to show that the game can in fact be won. The latter part can be solved by trying to make the integers zero – this has less to do with invariants, and more to do with manipulations of numbers.

Solution. Let a_0, a_1, a_2, a_3, a_4 be the numbers at the vertices in that order. Let $0 \le I \le 4$ be such that $I \equiv a_1 + 2a_2 + 3a_3 + 4a_4 \pmod 5$. We claim that I is invariant under any turn. This is because, any turn consists of changing I by $-mk - m(k+1) + 2m(k+3) \equiv 0 \pmod 5$. If we end up with 2011 at one vertex and zero at every other vertex then 2011 will have to be at the vertex I. It now remains to show that we can reach this point.

We first show that $(0,0,0,0,0)$ can be reached from $(0,0,5,-5,0)$. This can be acheived with the following turns:

$$(0,0,5,-5,0) \longrightarrow (2,0,1,-5,2) \longrightarrow (0,-2,1,-1,2) \\ \longrightarrow (-2,-2,2,0,2) \longrightarrow (0,-2,-2,0,4) \longrightarrow (0,0,0,0,0)\,.$$

Now suppose that $a_0 + a_1 + a_2 + a_3 + a_4 = 2011$, and assume without loss of generality that $I = 0$. Play a turn if necessary, and assume that $a_0 = 2011$. By a turn (m,k) we mean the turn where we add $2m$ to a_k and subtract m from a_{k-2} and a_{k+2} (with indices modulo 5). Play the following turns: $(a_1, 3)$ and $(-a_1, 4)$. Our configuration then looks like $(2011, 0, b_2, b_3, b_4)$. Play the turns $(-b_4, 4)$ and $(-b_4, 2)$ and $(b_4, 3)$ to get a configuration $(2011, 0, c_3, c_4, 0)$. Since the sum of the five numbers is always 2011 it follows that $c_3 = -c_4$. Moreover, since $I = 0$ is an invariant it follows that c_3 is divisible by 5. We have show that $(0,0,5,-5,0)$ can be brought to $(0,0,0,0,0)$, so it follows that we can reach $(2011,0,0,0,0)$ from $(2011,0,c_3,c_4,0)$. $\square$

Problem 3.3. There are 60 girls sitting around in a circle. One of them is initially given M chocolates. In each move, every girl who has at least 3 chocolates gives one chocolate each to both her neighbours. This procedure stops either if all the girls have a chocolate or if no girl has more than 2 chocolates. Find the minimum value of M for which every girl gets a chocolate.

Discussion. One of the first things to notice is that *there are only finitely many* configurations. In such cases it is worthwhile to analyse if there is a loop. Obviously, if we there is a semi-invariant associated with each configuration such that it (strictly) decreases (or increases) with every move then there cannot be any loop. Suppose that there cannot be a loop, then the procedure will stop. In that case, if $M \ge 119$ then it is not hard to see (by Pigeon-Hole-Principle) that the procedure cannot stop because of the second condition.

Note that if a girl gets a chocolate at any time, then she will have at least one chocolate thereafter. Since there is symmetry in the given problem it is therefore enough to that girl "opposite" to G_0, the girl to whom the chocolates were given initially, gets a chocolate. Also, to show that there is no loop one can consider 59 girls sitting in a line (instead of a circle).

Another important thing to observe here is that the when a girl gives chocolates to her neighbours the chocolates *move away* from her. This hints that the (sum of all) distances of chocolates might be a good choice for a semi-invariant. When talking about distances, there is a choice we need to make – the choice of the origin, or the choice of the girl from whom the distances are to be measured. In this case, the natural choice would be G_0 (or the girl opposite to her). With this idea, if we naively construct the semi-invariant $I(C) = \sum_i |a_i|$ where a_i is the distance from G_0 of the i-th chocolate, then we realise that $I(C)$ is monotonically (but not strictly) increasing. That is because, if a girl gives a chocolate each to her neighbour then the semi-invariant does not increase unless that girl is G_0.

It is good to keep in mind that when we talk about distances it is often useful to consider the squares of the distances rather than their absolute value. In this situation as well, if $J(C) = \sum_i a_i^2$ then it follows easily that $J(C)$ increases strictly with every move. This is enough to conclude that $M \geq 119$.

Solution. We shall only give here half the solution. That is, we shall only prove that if $M \geq 119$ then every girl gets a chocolate. The other part of the solution (which is not directly related to any semi-invariant calculation) is left as an exercise to the reader.

We shal denote by G_0 the girl to whom the chocolates were given initially. It is clear that if a girl gets a chocolate after some moves, then she will have a chocolate with her thereafter. Also it is clear that if the girl G^0 sitting diagonally opposite to G_0 gets a chocolate then everyone gets a chocolate.

Number the chocolates $1, 2, \ldots, M$. By a configuration, we mean a distribution of chocolates among the girls. So in the initial configuration C_0 one girl, say G_0, has 60 chocolates, and the rest have none. In a configuration C, we denote by $a_i(C)$ the distance of the i-th chocolate from G_0. Here the distance between two girls is taken to be 1. Define

$$J(C) = \sum_{i=1}^{M} a_i(C)^2 .$$

If there is a move that takes C to C', then by definition it follows that G^0 does not have any chocolate in C. Consider a girl who had more than two chocolates in C. Let her distance from G_0 be d. Then the change in $J(C)$ because of her action of distribution of chocolates is $-2d^2+(d-1)^2+(d+1)^2 = 1$. This proves that $J(C') > J(C)$.

Suppose that $M \geq 119$. If G^0 does not have a chocolate with her, then b Pigeon-Hole-Principle it follows that at least one of the other girls has at least three chocolates, and therefore the procedure continues (and thus increasing the semi-invariant). But the semi-invariant is bounded above by $M \cdot 29^2$, and hence it follows that all the girls get a chocolate. □

Problem 3.4. There are $n \geq 1$ bins $B_1, B_2, \ldots, B_n$ and there are a total of n balls in them. A *move* consists of one of the following: (a) if B_1 has at least two balls, then we can move one ball to B_2; (b) if B_n has at least two balls, then we can move one ball to B_{n-1}; or (c) if B_i has at least two balls for some $2 \leq i \leq n-1$, then we can move one ball to B_{i-1} and one to B_{i+1}. Show that starting from any initial configurations any sequence of moves will lead to the configuration in which all the bins have exactly one ball each.

Discussion. Let us first fix some notation. For $1 \leq i \leq n$, let a_i denote the number of balls in bin B_i, so we have $a_1 + a_2 + \cdots + a_n = n$. We can blindly try the kind of semi-invariants we have encountered so far, for example, $\sum a_i^2$. It is easy to see that this quantity does not satisfy any sort of monotonicity.

Let us look at the semi-invariant that we want from a slightly different perspective. We want it to measure how far we are from the desired configuration. In other words, we want to measure *how far the i-th ball is from the i-th bin.* Obviously, this measure is trivial in case of the desired configuration. Now let us see how we can calculate this measure. Since there are a_1 balls in the first bin, for $i = 1, 2, \ldots, a_1$, the distance of i-th ball from the i-th bin is $(i-1)^2$. We have taken the square so as to avoid the difficulty in dealing with the absolute values. For $i = a_1+1, \ldots, a_1+a_2$, the measure equals $(i-2)^2$ and so on. We will leave it to the reader to check that we eventually get that the sum of all these measures is

$$I(a_1, a_2, \ldots, a_n) = \sum_{k=1}^{n} (s_k - k)^2 \,,$$

where $s_k = a_1 + a_2 + \ldots + a_k$.

This semi-invariant is monotonically decreasing, but it fails to decrease strictly in one particular case. This failure can be set right if we assign a weight.

After the solution based on the above semi-invariant idea, we shall also give an alternate solution purely based on induction. We give this solution for two reasons. First of all, the semi-invariant used in the solution below looks bit tricky! And secondly, the induction also provides a bit more flexibility, in the sense that the solution is same for a simple generalization stated at the end.

We want to stress that induction is more powerful than what we think, but we may have to deal with it carefully at times. This problem is a nice example where a careful application of nested induction leads to an alternate solution. There are three quantities, namely r, c_r and s, on which we apply induction, but every step is quite simple. What makes the solution a bit complicated is the nested property.

Solution. For $1 \le k \le n$, let a_k denote the number of balls in bin B_k and let $s_k = a_1 + a_2 + \cdots + a_k$. Let $I(a_1, a_2, \ldots, a_n) = \sum_{k=1}^{n} (s_k - \theta k)^2$, where $0 < \theta < 1$ is a quantity we shall choose later. We then have

$$\begin{aligned} I(a_1, a_2, \ldots, a_n) - I(a_1 - 1, a_2 + 1, \ldots, a_n) &= 2(a_1 - \theta) - 1 > 0 \,, \\ I(a_1, a_2, \ldots, a_n) - I(a_1, \ldots, a_{k-1} + 1, a_k - 2, a_{k+1} + 1, \ldots, a_n) &= 2a_k - 2 - 2\theta > 0 \,, \\ I(a_1, a_2, \ldots, a_n) - I(a_1, \ldots, a_{n-1} + 1, a_n - 1) &= 2\theta(n-1) - (2n-3) \,, \end{aligned}$$

where the last quantity is positive provided $\theta > (2n-3)/(2n-2)$. Thus choosing such a θ we get that the value of I decreases with any move. Since there are only finitely many possible configurations it follows that the process should terminate. It remains to show that the minimum value of I is achieved at the desired configuration. This is true because $(s_k - \theta k)^2 - (k - \theta k)^2 = (s_k - k)(s_k - k + 2k(1-\theta)) \ge 0$ for all $k = 1, 2, \ldots, n$, and the equality holds if and only if $s_k = k$. $\square$

Alternate solution. Let t_k, for $1 \le k \le n$, denote the move in (a) if $k = 1$, move in (b) if $k = n$ or the move in (c) with $k = i$.

Suppose that we do not have the desired configuration. Let r be the smallest index such that B_r has at least two balls. We shall induct on $n - r$. The base case is when $r = n$. Let c_n denote the number of balls in B_n. We induct again on c_n. The base case now is when $c_n = 2$, and in this case let $s < n$ be the unique index such that B_s does not contain any ball. We induct (for the third time!) on $n-1-s$. If $s = n-1$ then applying t_n will give us the desired configuration. If $s < n-1$ then apply $t_n, t_{n-1}, \ldots, t_{s+1}$ to get a configuration in which $c_n = 2$ and B_{s+1} does not contain any ball in it, and hence we are done by induction on $n-1-s$. This completes the base case $c_n = 2$.

We now consider the (base!) case $r = n$ and $c_n > 3$. Let s be the largest index such that B_s does not contain any ball in it. We again induct on $n-1-s$. As before, if $s = n-1$ then applying

t_n will give us a configuration in which c_n is smaller and we are then done by induction on c_n. If $s < n-1$ then applying $t_n, t_{n-1}, \ldots, t_{s+1}$ will give us a configuration in which B_{s+1} does not contain any ball, and thus we are done by induction on $n-1-s$. This completes the base case $r = n$.

Now suppose that every configuration with $r > R$ can be brought into the desired configuration. Consider a configuration with $r = R$ and let $c_r \geq 2$ denote the number of balls in B_r. We apply induction on c_r. Consider the base case $c_r = 2$. If there is a bin B_i, with $i < R$, that does not contain any ball in it then let s denote the largest of such indices. Otherwise let $s = 0$. Applying $t_R, t_{R-1}, \ldots, t_{s+1}$ will then result in either the desired configuration or a configuration with $r > R$. This completes the proof of the base case $c_r = 2$.

Now assume that $c_r > 2$. Again let s be as in the case $c_r = 2$. In this case, applying $t_R, t_{R-1}, \ldots, t_{s+1}$ will reduce c_r and hence we are done by induction.

The solution is now complete. □

And here is a simple generalization of the previous problem.

Problem 3.5. Let $n, a_1, a_2, \ldots, a_n$ be positive integers and let $M = a_1 + a_2 + \ldots + a_n$. Consider n bins $B_1, B_2, \ldots, B_n$ which contain a total of M balls in them. A *move* consists of one of the following: (a) if B_1 has at least $a_1 + 1$ balls, then we can move one ball to B_2; (b) if B_n has at least $a_n + 1$ balls, then we can move one ball to B_{n-1}; or (c) if B_i has at least $a_i + 1$ balls for some $2 \leq i \leq n-1$, then we can move one ball to B_{i-1} and one to B_{i+1}. Show that starting from any initial configurations we can make appropriate moves to get a configuration in which, for all $1 \leq i \leq n$, B_i contains exactly a_i balls.

Problem 3.6 (IMO Shortlist 1996)**.** A finite number of coins are placed on an infinite row of squares. A sequence of moves is performed as follows: at each stage a square containing more than one coin is chosen. Two coins are taken from this square; one of them is placed on the square immediately to the left while the other is placed on the square immediately to the right of the chosen square. The sequence terminates if at some point there is at most one coin on each square. Given some initial configuration, show that any legal sequence of moves will terminate.

Remark. In fact, it is also true that any legal sequence of moves will terminate after same number of moves and in the same configuration.

Discussion. This problem is a variation of Problem 3.4. There are two differences: we now have an infinite row of squares instead of finitely many bins; and, we also have to show that the number of steps is same. Since there are only finite number of coins, we should be able to restrict ourselves to a finite set of rows. For this we have to show that the coins cannot go beyond certain squares. Intuitively, if S_k is the left-most square with a coin in it, then the squares to the left of it can never have all the coins in them. We shall use this in the solution below and show that only finitely many squares need to be considered.

We notice the similarity of this problem with Problem 3.2. A move at square S_k is symmetric at the square. With a_k denoting the number of coins in square S_k, we see that $\sum a_k$ will be an invariant, and moreover the weighted sum $\sum k a_k$ will also be an invariant (exactly like in the solution to Problem 3.2). We are however looking for a semi-invariant. An immediate thought would be to assign different weights (that almost balance each other out). Since $(k-1)^2 + (k+1)^2 - 2k^2 = 2$, the semi-invariant $\sum k^2 a_k$ should be a good choice.

Solution. Suppose that there are n coins. Let S_i denote the squares with i ranging over the set of all integers and let a_i denote the number of coins in S_i. A move means choosing an integer i with $a_i \geq 2$ and replacing a_{i-1}, a_i, a_{i+1} with $a_{i-1}+1, a_i-2, a_{i+1}+1$ respectively. Suppose that S_k is the left-most square that has a coin in the initial configuration. Then we notice that $a_k + a_{k-1} + \cdots \geq 1$ at any point of time. This is quite evident. We now prove by induction that for any $i \geq 0$ we have $a_{k-i} + a_{k-i+1} + \cdots \geq i+1$. The base case $i = 0$ is what we just noticed. Now consider the general case of $i > 0$ (assuming the statement for $i-1$). If the inequality does not hold at some point of time, then consider the move because of which this fails. This move has to happen at the square S_{k-i} which will then have to have at least two coins. We then get that $a_{k-i+1} + a_{k-i+2} + \cdots < i$, a contradiction to the induction hypothesis. We have thus proved that the coins cannot move to the left of S_{k-n+1}. Similarly, we can give a bound on the right-hand side as well, so we only have to consider finitely many squares.

Let $S_1, S_2, \ldots, S_N$ be the squares to be considered. Let $I(a_1, a_2, \ldots, a_N) = \sum_{k=1}^{N} k^2 a_k$. Then

$$I(a_1, a_2, \ldots, a_N) - I(a_1, \ldots, a_{k-1}+1, a_k-2, a_{k+1}+1, \ldots, a_N) = 2a_k - 2 > 0\,.$$

This proves that the any sequence of moves will terminate. [2]

□

4 In a different direction

The last problem we shall consider here is of a different nature than the ones discussed above. We have seen problems where our goal is to get to a final configuration. The problem below is to prove just the opposite – that there is no final configuration, but instead there is a repetition.

Problem 4.1 (IMO 1993, Problem 6)**.** Let $n > 1$ be an integer. In a circular arrangement of n lamps $L_0, L_1, \ldots, L_{n-1}$, each of which can be either ON or OFF, we start with the situation where all lamps are ON, and then carry out a sequence of steps $Step_0, Step_1, \ldots$. If L_{j-1} (with j taken modulo n) is ON then $Step_j$ changes the state of L_j (it goes from ON to OFF, or vice versa), but does not change the state of any other lamp. If L_{j-1} is OFF then $Step_j$ does not change anything at all. Show that:

(a) There is a positive integer $M(n)$ such that after $M(n)$ steps all lamps are ON again.

(b) If $n = 2^k$ for some positive integer k then all the lamps are ON after $n^2 - 1$ steps.

(c) If $n = 2^k + 1$ for some positive integer k then all the lamps are ON after $n^2 - n + 1$ steps.

Discussion. The key idea in such problems where we have to look for repetition is to consider a *backward* move. If there is a unique backward move, then it follows by the finiteness of the number of configurations that they will have to repeat after a while. In terms of graph theory, this is same as saying that if the degree of every vertex is exactly two (one for the forward move and one for the backward move) then it comprises of disjoint cycles.

It is time to again mention two things: examples and induction! For the second part, it is quite evident how the configuration changes if we work out couple of examples. It is then natural to

[2] We note that $\sum_{k=1}^{N} s_k^2$, with $s_k = a_1 + a_2 + \cdots + a_k$, is another strictly increasing semi-invariant.

consider a solution using induction. Working out examples will also help us see the similarities between the configurations in the second and third part.

We shall also briefly mention two alternative solutions using polynomials and generating functions. These ideas differ very much from our current topic, so we shall keep the arguments to bare minimal.

Solution. To make some of the notations simpler, we shall assume that we begin with $Step_1$ rather than $Step_0$. Note that this does not change any of the statements of the problem since we can always renumber the lamps and get the exact statement as in the problem.

(a) Write $(j, (a_0, a_1, \ldots, a_{n-1}))$ for a configuration of the lights, with each a_i being 0 if L_i is off or 1 otherwise, and j indicating that we have just finished $Step_j$, so the initial configuration is $(-1, (1, 1, \ldots, 1))$. Obviously, j is written modulo n. Clearly, there are only finitely many such configurations. This proves that the configurations repeat after a while. We note that $(j-1, (a_0, a_1, \ldots, a_{j-1}, a_j + a_{j-1}, a_{j+1}, \ldots, a_{n-1}))$ is the unique predecessor of $(j, (a_1, \ldots, a_n))$, and hence it follows that the initial configuration repeats.

(b) Suppose that $n = 2^k$. We shall show the following by induction:

- L_{n-1} is OFF after $Step_{n-1}$ and remains off until $Step_{n^2-1}$;
- L_0 is ON, and all the other lamps are OFF after $n^2 - n$ steps;
- all the lamps are ON after $n^2 - 1$ steps.

The statement is obvious for the base case $k = 1$. Suppose that $k > 1$ and that the statement holds if we replace k with $k - 1$. Let $m = 2^{k-1}$. The configuration after $n - 1$ steps is

$$(-1, \underbrace{1010\cdots10}_{m}\ \underbrace{1010\cdots10}_{m}).$$

Thanks to the first statement in the induction, it follows that after $2(m^2 - m)$ steps the configuration is

$$(0, \underbrace{1000\cdots00}_{m}\ \underbrace{1000\cdots00}_{m}),$$

and thus after $2m^2 - 1$ steps we will have

$$(-1, \underbrace{1111\cdots11}_{m}\ \underbrace{0000\cdots00}_{m}).$$

Another n steps will result in

$$(-1, \underbrace{1010\cdots10}_{m}\ \underbrace{0000\cdots00}_{m}).$$

Again from the first statement in the induction, it follows that after $n^2 - n$ steps we will have

$$(0, \underbrace{1000\cdots00}_{m}\ \underbrace{0000\cdots00}_{m}),$$

and hence all the lights will be ON after $n^2 - 1$ steps.

(c) The configuration after two steps is $(2, 1011\cdots 11)$ which is nothing but $(0, 1111\cdots 110)$ but shifted by 2. We shall consider this rotated configuration henceforth (and add 2 in the end). After another $2^k - 1$ steps we get

$$(2^k - 1, \underbrace{1010\cdots 10}_{2^k}0)\,.$$

From the previous part it follows the last two lamps remain off until we reach the configuration $(0, 100\cdots 00)$. We further note that it takes $(2^k - 1)(2^k + 1) = n^2 - 2n$ steps to reach this. After another $n - 1$ steps we will have all the lights ON, and counting the first two steps, we have thus performed exactly $n^2 - n + 1$ steps.

□

Alternate solution. There is a clever, but less intuitive solution. Consider the polynomial $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1}$. Then the corresponding polynomial after $Step_0$ and an index shift by 1 is $Q(x) = a_{n-1} + (a_0 + a_{n-1})x + a_1x^2 + \cdots + a_{n-2}x^{n-1}$. We note that $Q(x) \equiv xP(x) \pmod{x^n + x^{n-1} + 1}$ (where we consider all the polynomails with coefficients modulo 2). The corresponding polynomial after k steps (and with the index shifted by k) is therefore $x^kP(x) \pmod{x^n + x^{n-1} + 1}$. Hence it is enough to show:

(a) $x^{M(n)} \equiv 1 \pmod{x^n + x^{n-1} + 1}$ for some integer $M(n)$.

(b) If $n = 2^k$ then $x^{n^2-1} \equiv 1 \pmod{x^n + x^{n-1} + 1}$.

(c) If $n = 2^k + 1$ then $x^{n^2-n+1} \equiv 1 \pmod{x^n + x^{n-1} + 1}$.

For part (a) we note that there are only finitely many residue classes modulo $x^n + x^{n-1} + 1$ and hence there are integers $k > l$ such that $x^k \equiv x^l \pmod{x^n + x^{n-1} + 1}$. This proves that $x^{k-l} \equiv 1 \pmod{x^n + x^{n-1} + 1}$.

For part (b) we note that $x^{n^2} \equiv (x^{n-1} + 1)^n \equiv x^{n^2-n} + 1 \pmod{x^n + x^{n-1} + 1}$ since $\binom{n}{k}$ is even for all $1 \le k \le n - 1$. On the other hand $x^{n^2} = x^{n^2-n} \cdot x^n \equiv x^{n^2-1} + x^{n^2-n} \pmod{x^n + x^{n-1} + 1}$. This proves the required equivalence.

Proof of part (c) is similar to that of part (b). In this case we have $x^{n^2-1} \equiv x^{n^2-n} + x^{n-1} \pmod{x^n + x^{n-1} + 1}$ and then the required equivalence follows. □

Alternate solution. The above solution can also be seen in a slightly different way using generating function. Let a_{i+rn} equal 0 if the lamp L_i is OFF after it was operated for r times (i.e., after $i + (r - 1)n + 1$ steps) and equal 1 otherwise. Then we have $a_{j+n} = a_j + a_{j+n-1}$ for all $j \ge 0$ and $a_0 = a_1 = \cdots = a_{n-1} = 1$. If we let $G(x) = \sum a_jx^j$ it then follows that $G(x) = 1/(1 - x - x^n)$ (assuming that all the coefficients are considered modulo 2). This leads to a solution. □

5 Conclusion

We have seen a few different kind of semi-invariants in this section. One should always look for invariants, which might be at times completely useless for the solution, but will give an idea as to what semi-invariant we should look for. It is good to keep in mind some of the standard semi-invariants one can look for: $\sum a_k, \sum a_k^2, \sum ka_k, \sum k^2 a_k, \sum (a_{k-1} + a_k)^2, \sum s_k \ldots$, and also $\sum 1/a_k, \sum 1/a_k^2, \ldots$, or a combination of these.

Below are a few more problems. The last two problems, which can possibly be solved using semi-invariants, are of different nature.

6 Extra problems

Problem 6.1. The numbers $1, 2, \ldots, n$ are written on a blackboard. It is permitted to erase any two numbers a and b and write the new number $ab + a + b$. Find with proof, the number that can be on the blackboard after $n - 1$ such operations?

Problem 6.2. A circle is divided into six sectors. Then the numbers $1, 0, 1, 0, 0, 0$ are written into the sectors. You may increase two neighboring numbers by 1. Is it possible to equalize all numbers by a sequence of such steps?

Problem 6.3. Let n be a positive integer and $a_1, a_2, \ldots, a_n$ positive real numbers. We put these numbers into groups $G_1, G_2, \ldots G_r$, with $r \geq 1$, with each group being non-empty. For every a_i we let its relative rating to be the ratio of a_i to the sum of all the elements in the group containing a_i. A *move* consists of moving an element from one group to the other provided its relative rating increases. Show that we can only make finitely many moves.

Problem 6.4. On each square of a chessboard is a light bulb which has two states – on and off. A *move* consists of choosing a square and changing the state of the bulbs in that square and in the neighbouring squares (which share a side). Show that starting from any configuration we can make finitely many moves to reach a point where all the bulbs are switched off.

Problem 6.5. Let $a_1, a_2, ..., a_{100}$ be an ordered set of numbers. At each move it is allowed to choose any two numbers a_n, a_m and change them to $\frac{a_n^2}{a_m} - \frac{n}{m}(\frac{a_m^2}{a_n} - a_m)$ and $\frac{a_m^2}{a_n} - \frac{m}{n}(\frac{a_n^2}{a_m} - a_n)$ respectively. Determine if it is possible, starting with the set with $a_i = \frac{1}{5}$ for $i = 20, 40, 60, 80, 100$ and $a_i = 1$ otherwise, to obtain a set consisting of integers only.

Problem 6.6. We have $\frac{n(n+1)}{2}$ stones and they are divided into a few groups. In each move we take one stone from each group and form a new group with taken stones. Prove that after a few moves we will get groups with $1, 2, 3, \ldots, n$ stones (and after that position doesn't change).

Problem 6.7. Several boxes are arranged in a circle. Each box may be empty or may contain one or several chips. A move consists of taking all the chips from some box and distributing them one by one into subsequent boxes clockwise starting from the next box in the clockwise direction.

(a) Suppose that on each move (except for the first one) one must take the chips from the box where the last chip was placed on the previous move. Prove that after several moves the initial distribution of the chips among the boxes will reappear.

(b) Now, suppose that in each move one can take the chips from any box. Is it true that for every initial distribution of the chips you can get any possible distribution?

Problem 6.8 (IMO Shortlist 2009)**.** Consider 2009 cards, each having one gold side and one black side, lying on parallel on a long table. Initially all cards show their gold sides. Two player, standing by the same long side of the table, play a game with alternating moves. Each move consists of choosing a block of 50 consecutive cards, the leftmost of which is showing gold, and turning them all over, so those which showed gold now show black and vice versa. The last player who can make a legal move wins.

(a) Does the game necessarily end?

(b) Does there exist a winning strategy for the starting player?

Problem 6.9 (Korean National Mathematical Olympiad 2009, Problem 3)**.** There are 2008 white stones and one black stone in a row. A *move* means the following: select one black stone and change the colour of its neighbouring stone(s). Find all possible initial position of the black stone for which all the stones can be made black after finitely many moves.

Problem 6.10. There is $(2n-1) \times (2n-1)$ square and for every small square there is one arrow, up or down or right or left. A bug is on one of the small squares. It travels to a next square following the arrow. If bug leaves the square, the arrow of the square turn $\frac{\pi}{2}$ in counter clock wise.

Prove that this bug will be out of this big square before $2^{3n-1} \cdot (n-1)! - 3$ moves.