These are some notes (written by Tejaswi Navilarekallu) used at the International Mathematical Olympiad Training Camp (IMOTC) 2013 held in Mumbai during April-May, 2013.

1 Definition and some useful facts

Definition 1.1. Given four points A, B, C, D on a line, the *cross-ratio* of these points is

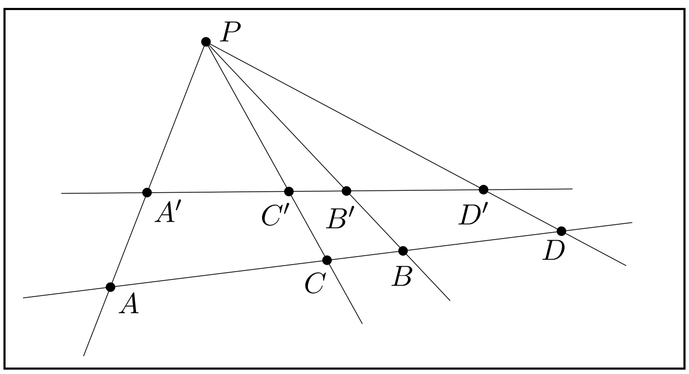
$$(A, B; C, D) = \frac{AC \cdot BD}{BC \cdot AD},$$

where the sign of a length is taken approproiately after fixing a particular orientation of the line. If $(A, B; C, D) = -1$ then the points are said to be in *harmonic range*, or we say that $(A, B; C, D)$ is a *harmonic bundle*. For any point P and a harmonic bundle $(A, B; C, D)$, the lines PA, PB, PC, PD are said to form a *harmonic pencil*.

Lemma 1.2. *Let A, B, C, D be points on a line, and P a point not on the line. Then*

$$(A, B; C, D) = \frac{\sin \angle APC}{\sin \angle APD} \cdot \frac{\sin \angle BPD}{\sin \angle BPC}.$$

Further, if a line not passing through P intersects PA, PB, PC, PD at A', B', C', D', resepctively, then $(A, B; C, D) = (A', B'; C', D')$. In particular, if $(A, B; C, D)$ is a harmonic bundle, then so is $(A', B'; C', D')$.

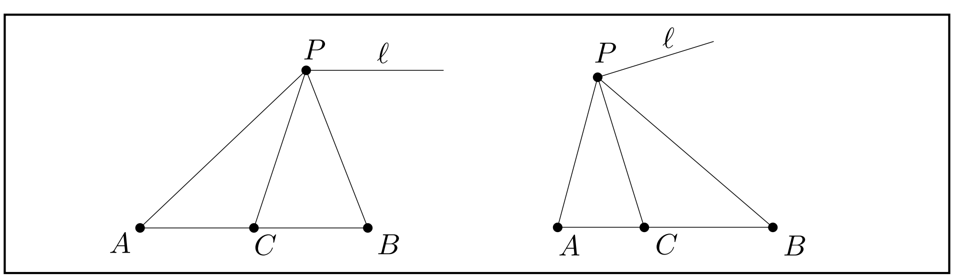


Lemma 1.3. *Let A, B, C points on a line, P a point not on that line and l a line through P. Then*

(a) if $AC = BC$ and l is parallel to the line AB then PA, PB, PC, l form a harmonic pencil;

(b) if the line PC bisects $\angle APB$ and l is perpendicular to line PC then PA, PB, PC, l form a harmonic pencil.

Proof. The statements follow directly from the definition and the previous lemma.



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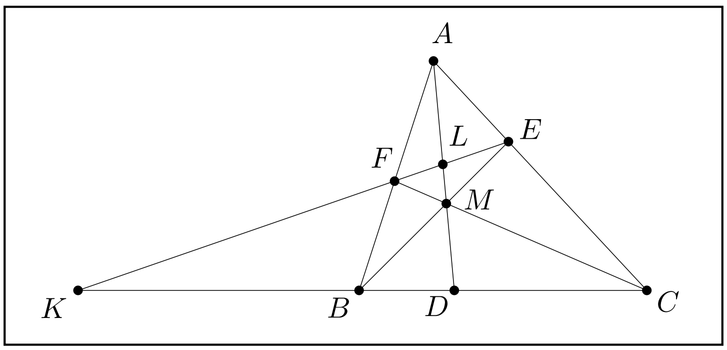
Lemma 1.4. *In a triangle ABC, points D, E, F lie on the lines BC, CA, AB respectively, such that AD, BE, CF are concurrent. The lines EF intersects BC at K. Then*

(a) $(B, C; D, K)$ is a harmonic bundle;

(b) AB, AC, AD, AK form a harmonic pencil;

(c) DE, DF, DA, DB form a harmonic pencil.

Proof. The first statement follows from Ceva's and Menelaus' theorems. The second statement follows from the first immediately. The third statement follows from the fact that $(E, F; L, K)$ is a harmonic bundle where L is the point of intersection of the lines AD and EF.



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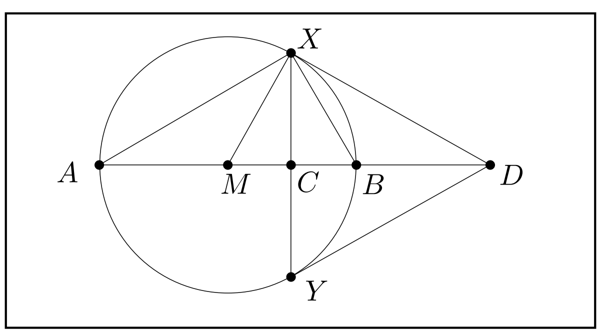
Lemma 1.5. *Let $(A, B; C, D)$ be a harmonic bundle, and M the midpoint AB. Then $MC \cdot MD = MA^2$ and $DC \cdot DM = DB \cdot DA$.*

Proof. Let X be a point on the circle with AB as diameter such that DX is tangent to that circle. Let C' be on AB such that XC' is perpendicular to AB. Then it is easy to verify that

$$\frac{\sin \angle AXC'}{\sin \angle BXC'} = \frac{\sin \angle AXD}{\sin \angle BXD},$$

and hence $C' = C$. Therefore $MC \cdot MD = MX^2 = MA^2$ and $DC \cdot DM = DX^2 = DB \cdot DA$.



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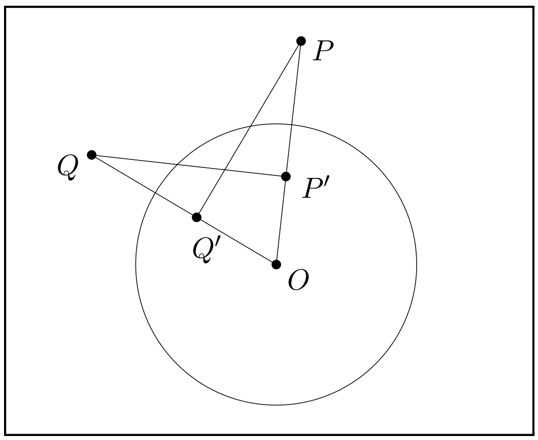
Definition 1.6. Given a circle Γ with center O and a point P different from O, let P' be the inversion of point P with respect to Γ, i.e., P' is a point on the ray OP such that $OP \cdot OP' = r^2$, where r is the radius of Γ. The line l perpendicular to OP and passing through P' is called the *polar* of point P with respect to Γ, and the point P is called the *pole* of l with respect to Γ.

Lemma 1.7. *Let Γ be a circle and P, Q points different from the center of Γ. If the polar of P with respect to Γ passes through Q then the polar of Q with respect to Γ passes through P.*

Proof. Let O and r denote the center and radius of Γ, respectively, and P', Q' the inversion of points P, Q, respectively, with respect to Γ. Then $OP \cdot OP' = OQ \cdot OQ' = r^2$. Therefore $QPQ'P'$ is a cyclic quadrilateral. Since Q lies on the polar of P we have $\angle QP'P = 90°$ and hence $\angle QQ'P = 90°$. This proves that P lies on the polar of Q.



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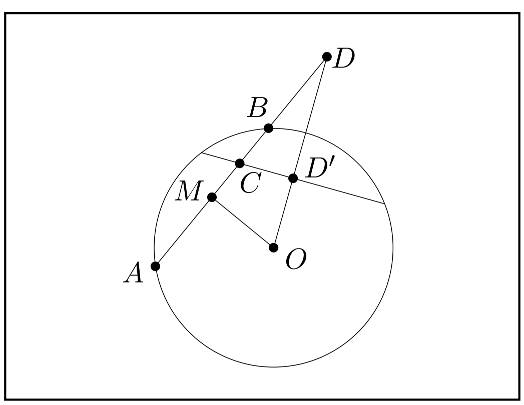
Lemma 1.8. *Let Γ be a circle and D a point not lying on Γ and different from its center. Let l be the polar of D with respect to Γ. A line drawn through D intersects l at C and Γ at A and B. Then $(A, B; C, D)$ is a harmonic bundle.*

Proof. Let O denote the center of Γ, D' the inversion of point D with respect to Γ and M the midpoint of AB. Then $OMCD'$ is a cyclic quadrilateral, so $DC \cdot DM = DD' \cdot DO = DB \cdot DA$ and hence $(A, B; C, D)$ is a harmonic bundle by Lemma 1.5.



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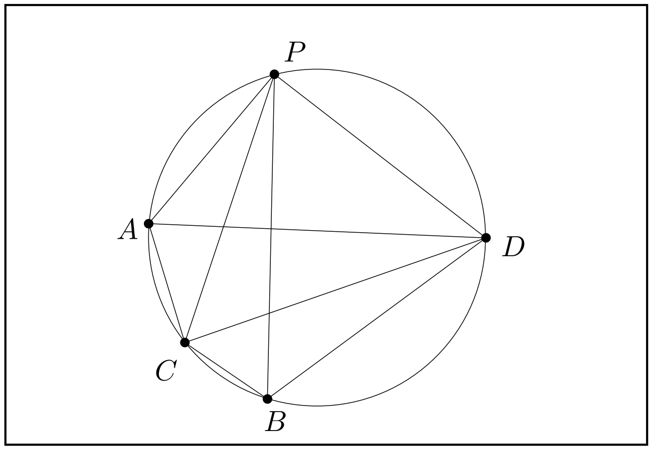
Lemma 1.9. *Let P, A, B, C, D be points on a circle. Then PA, PB, PC, PD form a harmonic pencil if and only if $\frac{AC}{BC} = \frac{AD}{BD}$. And in this case, the lines QA, QB, QC, QD form a harmonic pencil for any point Q on the circle.*

Proof. The first part is immediate from Sine rule and Lemma 1.2. The second part follows from the first part since the condition $\frac{AC}{BC} = \frac{AD}{BD}$ is independent of point P.



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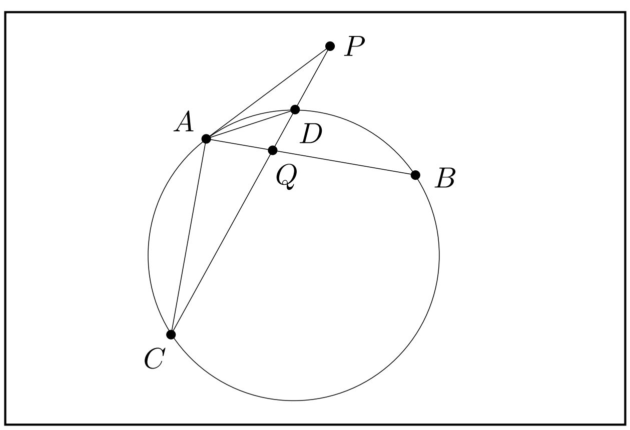
Definition 1.10. A cyclic quadrilateral $ACBD$ is called a *harmonic quadrilateral* if $\frac{AC}{BC} = \frac{AD}{BD}$.

Lemma 1.11. *Let Γ be a circle and P a point not lying on it and different from its center. Let A, B be points on Γ such that the line AB is the polar of P. A line through P intersects the circle at points C and D. Then $ACBD$ is a harmonic cyclic quadrilateral.*

Proof. Let Q be the point of intersection of line CD and AB. Then by Lemma 1.8 it follows that $(P, Q; C, D)$ is a harmonic bundle. Therefore AP, AQ, AC, AD form a harmonic pencil. Applying Lemma 1.9 with points A, A, B, C, D it follows that $ACBD$ is a harmonic quadarilateral.



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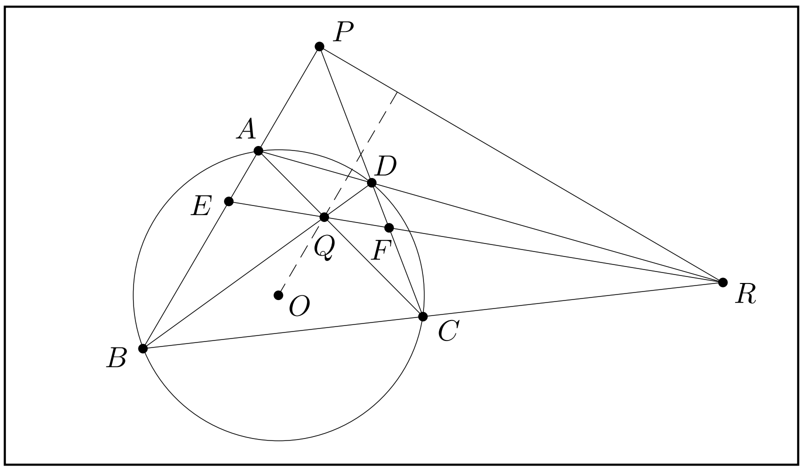
Lemma 1.12. *Drawn from a point P not lying on circle Γ are two lines intersecting the circle at A, B and C, D respectively. Then the point Q of intersection of the lines AC and BD lie on the polar of point P. Moreover, if the lines AD and BC intersect at point R then O is the orthocenter of triangle PQR, where O is the center of Γ.*

Proof. Let the lines QR intersect the lines AB and CD at E and F respectively. Then $(P, E; A, B)$ and $(P, F; C, D)$ are harmonic bundles by Lemma 1.4 and therefore E and F lie on the polar of point P by Lemma 1.8. Hence it also follows that PO is perpendicular to QR, and by similar argument QO is perpendicular to PR. This proves that O is the orthocenter of triangle PQR.



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2 Problems

Problem 2.1. (Sharygin, Selected problems 133) In a triangle ABC points D, E, F are taken on sides BC, CA, AB respectively such that AD, BE, CF concur at a point. If AD bisects $\angle EDF$ then prove that AD is perpendicular to BC.

Problem 2.2. (Sharygin, Selected problems 175) Let M and N denote the projections of the orthocenter of a triangle ABC on the internal and the external bisectors of $\angle B$. Prove that MN bisects AC.

Problem 2.3. (Sharygin, Selected problems 176) Given two points A and B on a circle Γ. Let C denote the intersection of tangents to Γ at A and B. The circle passing through C and touch AB at B intersects Γ again at M. Prove that AM bisects BC.

Problem 2.4. (Sharygin, Selected problems 177) Drawn to a circle from a point A, situated outside the circle, are two tangents AM and AN with M and N the points of tangency, and a secant intersecting the circle at K and L. An arbitrary line l is drawn parallel to AM. Let KM and LM intersect l at P and Q respectively. Prove that the line MN bisects the segment PQ.

Problem 2.5. (Sharygin, Selected problems 183) In a triangle ABC, constructed with the altitude BD as the diameter is a circle intersecting AB and AC at K and L, respectively. The lines touching the circle at K and L intersect at M. Prove that BM bisects AC.

Problem 2.6. (Sharygin, Selected problems 184) A straight line l is perpendicular to the line segment AB and passes through B. A circle centered on l passes through A and intersects l at points C and D. The tangents to the circles at points A and C intersect at N. Prove that the line DN bisects the line segment AB.

Problem 2.7. (Sharygin, Selected problems 185) Let N denote the intersection point of the tangents drawn to the circumcircle Γ of a triangle ABC at B and C. Let M be a point on Γ such that AM is parallel to BC. Let Γ intersect MN again at K. Prove that AK bisects BC.

Problem 2.8. (Sharygin, Selected problems 189) Let AB be the diameter of a semicircle and M a point on the diameter AB. Points C, D, E, F lie on the semicircle so that $\angle AMD = \angle EMB$ and $\angle CMA = \angle FMB$. Let P denote the intersection point of the lines CD and EF. Prove that the line PM is perpendicular to AB.

Problem 2.9. (Vietnam National Olympiad 2003, Problem 2) The circles C_1 and C_2 touch externally at M and the radius of C_2 is larger than that of C_1. Let A be a point on C_2 which does not lie on the line joining the centers of the circles, B and C points on C_1 such that AB and AC are tangent to C_1. The lines BM, CM intersect C_2 again at E, F respectively. Let D be the intersection of the tangent at A and the line EF. Show that the locus of D as A varies is a straight line.

Problem 2.10. (Vietnam National Olympiad 2009, Problem 3) Let A, B be two fixed points and C is a variable point on the plane such that $\angle ACB$ is a constant. Let D, E, F be the projections of the incenter I of triangle ABC to its sides BC, CA, AB, respectively. Denoted by M, N the intersections of AI, BI with DE, respectively. Prove that the length of the segment MN is constant and the circumcircle of triangle FMN always passes through a fixed point.

Problem 2.11. (IX Geometrical Olympiad in honour of I. F. Sharygin, 2013, Problem 21) Let A be a point inside a circle ω. One of two lines drawn through A intersects ω at points B and C, the second intersects at points D and E. The line passing through D and parallel to BC intersects ω for the second time at point F, and the line AF meets ω at point T. Let M be the common point of the lines ET and BC, and N the reflection of A across M. Prove that the circumcircle of triangle DEN passes through the midpoint of BC.

Problem 2.12. (IMO 1985, Problem 5) A circle with center O passes through the vertices A and C of the triangle ABC and intersects the segments AB and BC again at distinct points K and N respectively. Let M be the point of intersection of the circumcircles of triangles ABC and KBN (apart from B). Prove that $\angle OMB = 90^\circ$.

Problem 2.13. (IMO 1998, Problem 5) Let I be the incenter of triangle ABC. Let the incircle of ABC touch the sides BC, CA, AB at K, L, M respectively. The line through B parallel to MK intersects LM and LK at R and S respectively. Prove that $\angle RIS$ is acute.

Problem 2.14. (IMO 2004, Problem 5) In a convex quadrilateral $ABCD$ the diagonal BD does not bisect the angles ABC and CDA. The point P lies inside $ABCD$ and satisfies

$$\angle PBC = \angle DBA \quad \text{and} \quad \angle PDC = \angle BDA\,.$$

Prove that $ABCD$ is a cyclic quadrilateral if and only if $AP = CP$.

Problem 2.15. (IMO 2012, Problem 1) In the triangle ABC the point J is the center of the excircle opposite to A. This excircle is tangent to the side BC at M, and to the lines AB and AC at K and L respectively. The lines LM and BJ meet at F, and the lines KM and CJ meet at G. Let S be the point of intersection of the lines AF and BC, and let T be the point of intersection of the lines AG and BC. Prove that M is the midpoint of ST.

Problem 2.16. (IMO ShortList 1998, Problem G8) Let ABC be a triangle such that $\angle A = 90°$ and $\angle B < \angle C$. The tangent at A to the circumcircle ω of triangle ABC meets the line BC at D. Let E be the reflection of A in the line BC, let X be the foot of the perpendicular from A to BE, and let Y be the midpoint of the segment AX. Let the line BY intersect the circle ω again at Z.

Prove that the line BD is tangent to the circumcircle of triangle ADZ.

Problem 2.17. (IMO ShortList 2004, Problem G8) Given a cyclic quadrilateral $ABCD$, M is the midpoint of CD, E is the point of interesection of lines AC and BD, F is the point of intersection of lines AD and BC, and $N \neq M$ is a point on the circumcircle of triangle ABM such that $AN/BN = AM/BM$. Prove that E, F, N are collinear.

Problem 2.18. (IMO ShortList 2009, Problem G4) Given a cyclic quadrilateral $ABCD$, let the diagonals AC and BD meet at E and the lines AD and BC meet at F. The midpoints of AB and CD are G and H, respectively. Show that EF is tangent at E to the circle through the points E, G and H.

Problem 2.19. In a triangle ABC, a circle Γ is drawn with AH as diameter, where H is the orthocenter. Points P and Q are on Γ such that the lines BP and BQ are tangents to Γ. Prove that P, Q, C are collinear.

Problem 2.20. In a triangle ABC, let O be its circumcenter and P a point (different from O) on the circumcircle of triangle BOC such that OP is perpendicular to BC. Prove that the symmedian point of triangle ABC lies on the line AP.

Problem 2.21. The point D is the foot of perpendicular from A in triangle ABC, P a point on AD, F a point on AC. Lines BP and CP intersect AC and AB at M and N respectively; lines MN and AD intersect at Q; and, lines FQ and CN intersect at E. Prove that $\angle EDA = \angle FDA$.

Problem 2.22. Point M lies on diagonal BD of a parallelogram $ABCD$. Line AM intersects lines CD and BC at K and N respectively. Denote by Γ_1 the circle with M as center and MA as radius, and by Γ_2 the circumcircle of triangle CKN. If P and Q are the points of intersection of these two circles then prove that MP and MQ are tangents to Γ_2.