INMO-2001
Problems and Solutions

1. Let ABC be a triangle in which *no* angle is $90°$. For any point P in the plane of the triangle, let A_1, B_1, C_1 denote the reflections of P in the sides BC, CA, AB respectively. Prove the following statements:

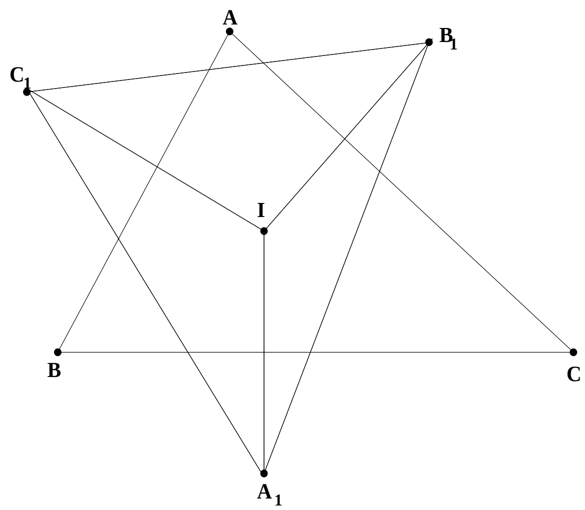
   (a) If P is the incentre or an excentre of ABC, then P is the circumcentre of $A_1B_1C_1$;

   (b) If P is the circumcentre of ABC, then P is the orthocentre of $A_1B_1C_1$;

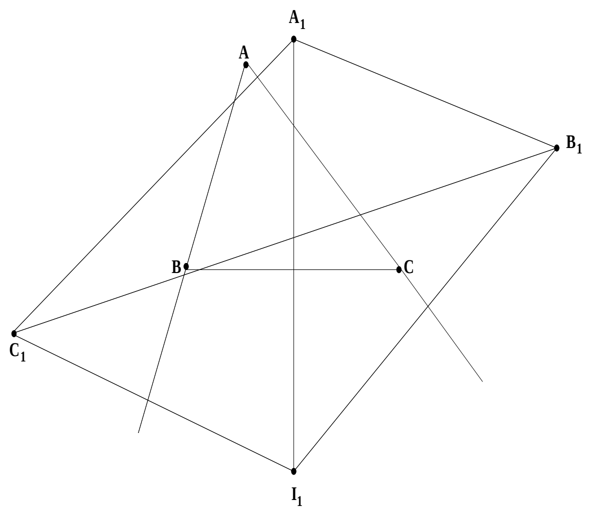
   (c) If P is the orthocentre of ABC, then P is either the incentre or an excentre of $A_1B_1C_1$.

Solution:

(a)



If $P = I$ is the incentre of triangle ABC, and r its inradius, then it is clear that $A_1I = B_1I = C_1I = 2r$. It follows that I is the circumcentre of $A_1B_1C_1$. On the otherhand if $P = I_1$ is the excentre of ABC opposite A and r_1 the corresponding ex-radius, then again we see that $A_1I_1 = B_1I_1 = C_1I_1 = 2r_1$. Thus I_1 is the circumcentre of $A_1B_1C_1$.



(b)

Let $P = O$ be the circumcentre of ABC. By definition, it follows that OA_1 bisects and is bisected by BC and so on. Let D, E, F be the mid-points of BC, CA, AB respectively. Then FE is parallel to BC. But E, F are also mid-points of OB_1, OC_1 and hence FE is parallel to B_1C_1 as well. We conclude that BC is parallel to B_1C_1. Since OA_1 is perpendicular to BC, it follows that OA_1 is perpendicular to B_1C_1. Similarly OB_1 is perpendicular to C_1A_1 and OC_1 is perpendicular to A_1B_1. These imply that O is the orthocentre of $A_1B_1C_1$. (This applies whether O is inside or outside ABC.)

(c)

let $P = H$, the orthocentre of ABC. We consider two possibilities; H falls inside ABC and H falls outside ABC.

Suppose H is inside ABC; this happens if ABC is an acute triangle. It is known that A_1, B_1, C_1 lie on the circumcircle of ABC. Thus $\angle C_1A_1A = \angle C_1CA = 90^\circ - A$. Similarly $\angle B_1A_1A = \angle B_1BA = 90^\circ - A$. These show that $\angle C_1A_1A = \angle B_1A_1A$. Thus A_1A is an internal bisector of $\angle C_1A_1B_1$. Similarly we can show that B_1 bisects $\angle A_1B_1C_1$ and C_1C bisects $\angle B_1C_1A_1$. Since A_1A, B_1B, C_1C concur at H, we conclude that H is the incentre of $A_1B_1C_1$.

OR If D, E, F are the feet of perpendiculars of A, B, C to the sides BC, CA, AB respectively, then we see that EF, FD, DE are respectively parallel to B_1C_1, C_1A_1, A_1B_1. This implies that $\angle C_1A_1H = \angle FDH = \angle ABE = 90^\circ - A$, as $BDHF$ is a cyclic quadrilateral. Similarly, we can show that $\angle B_1A_1H = 90^\circ - A$. It follows that A_1H is the internal bisector of $\angle C_1A_1B_1$. We can proceed as in the earlier case.

If H is outside ABC, the same proofs go through again, except that two of A_1H, B_1H, C_1H are external angle bisectors and one of these is an internal angle bisector. Thus H becomes an excentre of triangle $A_1B_1C_1$.

2. Show that the equation

$$x^2 + y^2 + z^2 = (x - y)(y - z)(z - x)$$

has infinitely many solutions in integers x, y, z.

Solution: We seek solutions (x, y, z) which are in arithmetic progression. Let us put $y - x = z - y = d > 0$ so that the equation reduces to the form

$$3y^2 + 2d^2 = 2d^3.$$

Thus we get $3y^2 = 2(d - 1)d^2$. We conclude that $2(d - 1)$ is 3 times a square. This is satisfied if $d - 1 = 6n^2$ for some n. Thus $d = 6n^2 + 1$ and $3y^2 = d^2 \cdot 2(6n^2)$ giving us $y^2 = 4d^2n^2$. Thus we can take $y = 2dn = 2n(6n^2 + 1)$. From this we obtain $x = y - d = (2n - 1)(6n^2 + 1)$, $z = y + d = (2n + 1)(6n^2 + 1)$. It is easily verified that

$$(x, y, z) = ((2n - 1)(6n^2 + 1), 2n(6n^2 + 1), (2n + 1)(6n^2 + 1)),$$

is indeed a solution for a fixed n and this gives an infinite set of solutions as n varies over natural numbers.

3. If a, b, c are positive real numbers such that $abc = 1$, prove that

$$a^{b+c}\, b^{c+a}\, c^{a+b} \leq 1.$$

Solution: Note that the inequality is symmetric in a, b, c so that we may assume that $a \geq b \geq c$. Since $abc = 1$, it follows that $a \geq 1$ and $c \leq 1$. Using $b = 1/ac$, we get

$$a^{b+c}\, b^{c+a}\, c^{a+b} = \frac{a^{b+c}c^{a+b}}{a^{c+a}c^{c+a}} = \frac{c^{b-c}}{a^{a-b}} \leq 1,$$

because $c \leq 1$, $b \geq c$, $a \geq 1$ and $a \geq b$.

4. Given any nine integers show that it is possible to choose, from among them, four integers a, b, c, d such that $a + b - c - d$ is divisible by 20. Further show that such a selection is not possible if we start with eight integers instead of nine.

Solution:

Suppose there are four numbers a, b, c, d among the given nine numbers which leave the same remainder modulo 20. Then $a + b \equiv c + d \pmod{20}$ and we are done.

If not, there are two possibilities:

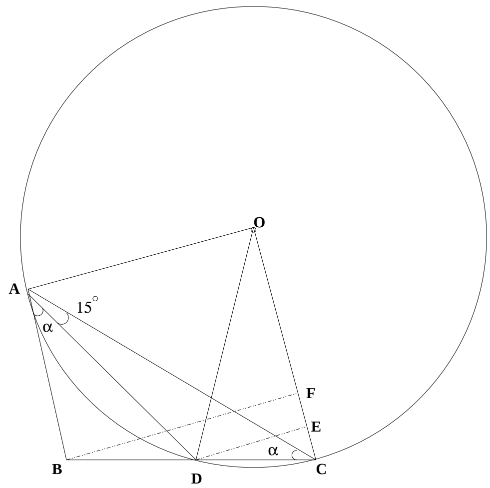
(1) We may have two disjoint pairs $\{a, c\}$ and $\{b, d\}$ obtained from the given nine numbers such that $a \equiv c \pmod{20}$ and $b \equiv d \pmod{20}$. In this case we get $a+b \equiv c+d$ (mod 20).

(2) Or else there are at most three numbers having the same remainder modulo 20 and the remaining six numbers leave distinct remainders which are also different from the first remainder (i.e., the remainder of the three numbers). Thus there are at least 7 disinct remainders modulo 20 that can be obtained from the given set of nine numbers. These 7 remainders give rise to $\binom{7}{2} = 21$ pairs of numbers. By pigeonhole principle, there must be two pairs $(r_1, r_2), (r_3, r_4)$ such that $r_1 + r_2 \equiv r_3 + r_4 \pmod{20}$. Going back we get four numbers a, b, c, d such that $a + b \equiv c + d \pmod{20}$.

If we take the numbers $0, 0, 0, 1, 2, 4, 7, 12$, we check that the result is not true for these eight numbers.

5. Let ABC be a triangle and D be the mid-point of side BC. Suppose $\angle DAB = \angle BCA$ and $\angle DAC = 15°$. Show that $\angle ADC$ is obtuse. Further, if O is the circumcentre of ADC, prove that triangle AOD is equilateral.

Solution:



Let α denote the equal angles $\angle BAD = \angle DCA$. Using sine rule in triangles DAB and DAC, we get

$$\frac{AD}{\sin B} = \frac{BD}{\sin \alpha}, \quad \frac{CD}{\sin 15°} = \frac{AD}{\sin \alpha}.$$

Eliminating α (using $BD = DC$ and $2\alpha + B + 15° = \pi$), we obtain $1 + \cos(B + 15°) = 2 \sin B \sin 15°$. But we know that $2 \sin B \sin 15° = \cos(B - 15°) - \cos(B + 15°)$. Putting $\beta = B - 15°$, we get a relation $1 + 2\cos(\beta + 30) = \cos \beta$. We write this in the form

$$(1 - \sqrt{3}) \cos \beta + \sin \beta = 1.$$

Since $\sin \beta \leq 1$, it follows that $(1 - \sqrt{3}) \cos \beta \geq 0$. We conclude that $\cos \beta \leq 0$ and hence that β is obtuse. So is angle B and hence $\angle ADC$.

We have the relation $(1-\sqrt{3})\cos\beta+\sin\beta=1$. If we set $x=\tan(\beta/2)$, then we get, using $\cos\beta=(1-x^2)/(1+x^2), \sin\beta=2x/(1+x^2)$,

$$(\sqrt{3}-2)x^2+2x-\sqrt{3}=0.$$

Solving for x, we obtain $x=1$ or $x=\sqrt{3}(2+\sqrt{3})$. If $x=\sqrt{3}(2+\sqrt{3})$, then $\tan(\beta/2)>2+\sqrt{3}=\tan 75°$ giving us $\beta>150°$. This forces that $B>165°$ and hence $B+A>165°+15°=180°$, a contradiction. thus $x=1$ giving us $\beta=\pi/2$. This gives $B=105°$ and hence $\alpha=30°$. Thus $\angle DAO=60°$. Since $OA=OD$, the result follows.

OR

Let m_a denote the median AD. Then we can compute

$$\cos\alpha=\frac{c^2+m_a^2-(a^2/4)}{2cm_a},\quad \sin\alpha=\frac{2\Delta}{cm_a},$$

where Δ denotes the area of triangle ABC. These two expressions give

$$\cot\alpha=\frac{c^2+m_a^2-(a^2/4)}{4\Delta}.$$

Similarly, we obtain

$$\cot\angle CAD=\frac{b^2+m_a^2-(a^2/4)}{4\Delta}.$$

Thus we get

$$\cot\alpha-\cot 15°=\frac{c^2-a^2}{4\Delta}.$$

Similarly we can also obtain

$$\cot B-\cot\alpha=\frac{c^2-a^2}{4\Delta},$$

giving us the relation

$$\cot B=2\cot\alpha-\cot 15°.$$

If B is acute then $2\cot\alpha>\cot 15°=2+\sqrt{3}>2\sqrt{3}$. It follows that $\cot\alpha>\sqrt{3}$. This implies that $\alpha<30°$ and hence

$$B=180°-2\alpha-15°>105°.$$

This contradiction forces that angle B is obtuse and consequently $\angle ADC$ is obtuse.

Since $\angle BAD=\alpha=\angle ACD$, the line AB is tangent to the circumcircle Γ of ADC at A. Hence OA is perpendicular to AB. Draw DE and BF perpendicular to AC, and join OD. Since $\angle DAC=15°$, we see that $\angle DOC=30°$ and hence $DE=OD/2$. But DE is parallel to BF and $BD=DC$ shows that $BF=2DE$. We conclude that

$BF = DO$. But $DO = AO$, both being radii of Γ. Thus $BF = AO$. Using right triangles BFO and BAO, we infer that $AB = OF$. We conclude that $ABFO$ is a rectangle. In particular $\angle AOF = 90°$. It follows that

$$\angle AOD = 90° - \angle DOC = 90° - 30° = 60°.$$

Since $OA = OD$, we conclude that AOD is equilateral.

OR

Note that triangles ABD and CBA are similar. Thus we have the ratios

$$\frac{AB}{BD} = \frac{CB}{BA}.$$

This reduces to $a^2 = 2c^2$ giving us $a = \sqrt{2}c$. This is equivalent to $\sin^2(\alpha + 15°) = 2\sin^2\alpha$. We write this in the form

$$\cos 15° + \cot\alpha \sin 15° = \sqrt{2}.$$

Solving for $\cot\alpha$, we get $\cot\alpha = \sqrt{3}$. We conclude that $\alpha = 30°$, and the result follows.

6. Let $\mathbf{R}$ denote the set of all real numbers. Find all functions $f : \mathbf{R} \to \mathbf{R}$ satisfying the condition

$$f(x + y) = f(x)f(y)f(xy)$$

for all x, y in $\mathbf{R}$.

Solution: Putting $x = 0, y = 0$, we get $f(0) = f(0)^3$ so that $f(0) = 0, 1$ or -1. If $f(0) = 0$, then taking $y = 0$ in the given equation, we obtain $f(x) = f(x)f(0)^2 = 0$ for all x.

Suppose $f(0) = 1$. Taking $y = -x$, we obtain

$$1 = f(0) = f(x - x) = f(x)f(-x)f(-x^2).$$

This shows that $f(x) \neq 0$ for any $x \in \mathbf{R}$. Taking $x = 1, y = x - 1$, we obtain

$$f(x) = f(1)f(x-1)^2 = f(1)\left[f(x)f(-x)f(-x)\right]^2.$$

Using $f(x) \neq 0$, we conclude that $1 = kf(x)(f(-x))^2$, where $k = f(1)(f(-1))^2$. Changing x to $-x$ here, we also infer that $1 = kf(-x)(f(x))^2$. Comparing these expressions we see that $f(-x) = f(x)$. It follows that $1 = kf(x)^3$. Thus $f(x)$ is constant for all x. Since $f(0) = 1$, we conclude that $f(x) = 1$ for all real x.

If $f(0) = -1$, a similar analysis shows that $f(x) = -1$ for all $x \in \mathbf{R}$. We can verify that each of these functions satisfies the given functional equation. Thus there are three solutions, all of them being constant functions.