# Regional Mathematical Olympiad 2015 (Mumbai region)
# 06 December, 2015

- **There are eight questions in this question paper. Answer all questions.**
- **Each of the questions 1,2,3 carries 10 points. Each of the questions 4,5,6,7,8 carries 14 points.**
- **Use of protractors, calculators, mobile phone is forbidden.**
- **Time allotted: 4 hours**

1. Let $ABCD$ be a convex quadrilateral with $AB = a$, $BC = b$, $CD = c$ and $DA = d$. Suppose
$$a^2 + b^2 + c^2 + d^2 = ab + bc + cd + da,$$
and the area of $ABCD$ is 60 square units. If the length of one of the diagonals is 30 units, determine the length of the other diagonal.

2. Determine the number of 3-digit numbers in base 10 having at least one 5 and at most one 3.

3. Let $P(x)$ be a non-constant polynomial whose coefficients are positive integers. If $P(n)$ divides $P(P(n) - 2015)$ for every natural number $n$, prove that $P(-2015) = 0$.

4. Find all three digit natural numbers of the form $(abc)_{10}$ such that $(abc)_{10}$, $(bca)_{10}$ and $(cab)_{10}$ are in geometric progression. (Here $(abc)_{10}$ is representation in base 10.)

5. Let $ABC$ be a right-angled triangle with $\angle B = 90^\circ$ and let $BD$ be the altitude from $B$ on to $AC$. Draw $DE \perp AB$ and $DF \perp BC$. Let $P$, $Q$, $R$ and $S$ be respectively the incentres of triangle $DFC$, $DBF$, $DEB$ and $DAE$. Suppose $S$, $R$, $Q$ are collinear. Prove that $P$, $Q$, $R$, $D$ lie on a circle.

6. Let $S = \{1, 2, \ldots, n\}$ and let $T$ be the set of all ordered triples of subsets of $S$, say $(A_1, A_2, A_3)$, such that $A_1 \cup A_2 \cup A_3 = S$. Determine, in terms of $n$,
$$\sum_{(A_1,A_2,A_3)\in T} |A_1 \cap A_2 \cap A_3|$$
where $|X|$ denotes the number of elements in the set $X$. (For example, if $S = \{1, 2, 3\}$ and $A_1 = \{1, 2\}$, $A_2 = \{2, 3\}$, $A_3 = \{3\}$ then one of the elements of $T$ is $(\{1, 2\}, \{2, 3\}, \{3\})$.)

7. Let $x$, $y$, $z$ be real numbers such that $x^2 + y^2 + z^2 - 2xyz = 1$. Prove that
$$(1 + x)(1 + y)(1 + z) \leq 4 + 4xyz.$$

8. The length of each side of a convex quadrilateral $ABCD$ is a positive integer. If the sum of the lengths of any three sides is divisible by the length of the remaining side then prove that some two sides of the quadrilateral have the same length.

**END OF QUESTION PAPER**