Regional Mathematical Olympiad 2014 (Mumbai region)

- **There are six questions in this question paper. Answer all questions.**
- **Each question carries 10 points.**
- **Use of protractors, calculators, mobile phone is forbidden.**
- **Time allotted: 3 hours**

1. Three positive real numbers a, b, c are such that $a^2 + 5b^2 + 4c^2 - 4ab - 4bc = 0$. Can a, b, c be the lengths of the sides of a triangle? Justify your answer.

2. The roots of the equation

$$x^3 - 3ax^2 + bx + 18c = 0$$

form a non-constant arithmetic progression and the roots of the equation

$$x^3 + bx^2 + x - c^3 = 0$$

form a non-constant geometric progression. Given that a, b, c are real numbers, find all positive integral values a and b.

3. Let ABC be an acute-angled triangle in which $\angle ABC$ is the largest angle. Let O be its circumcentre. The perpendicular bisectors of BC and AB meet AC at X and Y respectively. The internal bisectors of $\angle AXB$ and $\angle BYC$ meet AB and BC at D and E respectively. Prove that BO is perpendicular to AC if DE is parallel to AC.

4. A person moves in the $x - y$ plane moving along points with integer co-ordinates x and y only. When she is at point (x, y), she takes a step based on the following rules:

   (a) if $x + y$ is even she moves to either $(x + 1, y)$ or $(x + 1, y + 1)$;

   (b) if $x + y$ is odd she moves to either $(x, y + 1)$ or $(x + 1, y + 1)$.

   How many distinct paths can she take to go from $(0, 0)$ to $(8, 8)$ given that she took exactly three steps to the right $((x, y)$ to $(x + 1, y))$?

5. Let a, b, c be positive numbers such that

$$\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} \leq 1.$$

Prove that $(1 + a^2)(1 + b^2)(1 + c^2) \geq 125$. When does the equality hold?

6. Let D, E, F be the points of contact of the incircle of an acute-angled triangle ABC with BC, CA, AB respectively. Let I_1, I_2, I_3 be the incentres of the triangles AFE, BDF, CED, respectively. Prove that the lines I_1D, I_2E, I_3F are concurrent.

GOOD LUCK